JOURNAL OF APPROXIMATION THEORY 16, 347–365 (1976)

On the Convergence of Averaging Hermite Interpolators

M. A. BOTTO*

College of Cape Breton, Sydney Campus, P.O. Box 760, Sydney, N. S. B1P 6J1 Canada

Communicated by Oved Shisha

Received July 1, 1974

We investigate two sequences of polynomial operators, $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$, of degrees $2n-2$ and $2n-3$, respectively, defined by interpolatory conditions similar to those of the classical Hermite–Féjer interpolators $H_{2n-1}(f, x)$. If $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$ are based on the zeros of the jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, their convergence behaviour is similar to that of $H_{2n-1}(f, x)$. If they are based on the zeros of $(1-x^2)T_{n-2}(x)$, their convergence behaviour is better, in some sense, than that of $H_{2n-1}(f, x)$.

1. INTRODUCTION

Let $X_n = (x_{1n}, \ldots, x_{nn})$, $1 \geqslant x_{1n} > \cdots > x_{nn} \geqslant -1$, and $G_n = (g_{1n}, \ldots, g_{nn})$ denote the nth row of two triangular matrices X and G. For simplicity, we shall often write x_k, g_k for x_{kn}, g_{kn}. Also, set

$$\begin{aligned} w(x) &\equiv w_n(x) = (x-x_1)\cdots(x-x_n), \\ l_k(x) &\equiv l_{kn}(x) = w(x)/(x-x_k)\, w'(x_k), \end{aligned} \tag{1.1}$$

$$\begin{aligned} h_k(x) &\equiv h_{kn}(x) = \left(1-(x-x_k)\frac{w''(x_k)}{w'(x_k)}\right) l_k^2(x), \\ h_k^*(x) &\equiv h_{kn}^*(x) = (x-x_k)\, l_k^2(x), \end{aligned} \tag{1.2}$$

and let $f \in C[-1, 1]$ be given. The $(2n-1)$-degree polynomial

$$H_{2n-1}^G(f, x) = \sum_{k=1}^n f(x_k)\, h_k(x) + \sum_{k=1}^n g_k h_k^*(x) \tag{1.3}$$

is the well-known Hermite polynomial based on the nodes X_n, with derivatives at the nodes equal to $g_1, \ldots, g_n$. When G is identically zero, (1.3) is

* Work for this paper was done at the Univ. of Alberta, Edmonton, Alberta, Canada, as part of a Ph.D. thesis directed by Professor A. Sharma. Work supported by Canada Council Grant W73 4793.

better known as the Hermite–Féjer polynomial, which we shall simply denote $H_{2n-1}(f, x)$.

The convergence behaviour of the sequence of polynomials (1.3), as n tends to infinity, has been investigated for a variety of matrices X and G. It is known, for instance, that when $w(x) = P_n^{(\alpha,\beta)}(x)$, where $P_n^{(\alpha,\beta)}(x)$ denotes the nth Jacobi polynomial with $\alpha, \beta > -1$, and the vectors G_n are bounded, the sequence $H_{2n-1}^G(f, x)$ converges to $f(x)$ on $(-1, 1)$, uniformly on any closed subinterval [3, 6, 8].

On the other hand, Berman has shown [1, 2] that, when $w(x) = (1 - x^2)\, T_{n-2}(x)$, where $T_N(x) = \cos(N \arccos x)$, the Hermite–Féjer polynomials $H_{2n-1}(f, x)$ actually diverge on $(-1, 1)$ if $f(x) = x$, $|x|$, or x^2.

In the light of the above results, it would be interesting to see whether, by an appropriate choice of the matrix G, it is possible to obtain a sequence of polynomials that converges to $f(x)$ for every $f \in C[-1, 1]$, in both the cases $w(x) = P_n^{(\alpha,\beta)}(x)$ and $w(x) = (1 - x^2)\, T_{n-2}(x)$.

As a partial result in that direction, we construct two polynomial sequences $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$ (Section 2), with the following properties. If $w(x) = P_n^{(\alpha,\beta)}(x)$ (with $\alpha, \beta > -1$), then both $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$ converge to $f(x)$ on $(-1, 1)$, uniformly on every closed subinterval (Theorem 3.1). If $w(x) = (1 - x^2)\, T_{n-2}(x)$, then the uniform convergence class of $H_{2n-1}(f, x)$ is strictly contained in that of $H_{2n-2}(A_1, f; x)$, which in turn is strictly contained in that of $H_{2n-3}(A_2, f; x)$ (Theorem 7.1). Such convergence classes are partially characterized in Sections 5 and 6 (Theorems 5.1 and 6.1–6.4).

2. Definition of $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$

Let m be an integer less than n, and let $A_m(z) = \sum_{i=0}^{m} a_i z^i$ be a polynomial with positive roots. It has been proved [5, Theorem 2.1] that, for every $f(x)$, there exists a unique polynomial $H(x) = H_{2n-1-m}(A_m, f; x)$ of degree $2n - 1 - m$ or less, satisfying the conditions

$$H(x_k) = f(x_k),\ k = 1,\dots, n,\ \sum_{i=0}^{m} a_i H'(x_{i+j}) = 0, \qquad j = 1,\dots, n - m. \tag{2.1}$$

Such a polynomial may be called an *averaging Hermite interpolator* of $f(x)$, on account of the conditions on the derivatives. We shall concern ourselves only with the cases of the polynomials

$$A_1(z) = 1 - z (m = 1), \qquad A_2(z) = (1 - z)^2 (m = 2). \tag{2.2}$$

Note that if $A_0(z) \equiv 1$, then $H_{2n-1}(A_0, f; x) \equiv H_{2n-1}(f, x)$.

To find an explicit expression for $H_{2n-1-m}(A_m, f; x)$ in the two cases (2.2), let $s_n = x_1 + \cdots + x_n$, and set

$$J_n = \sum_{k=1}^{n} \frac{1}{w'(x_k)^2}, \qquad J_n{}^* = \sum_{k=1}^{n} \frac{2s_n - x_k}{w'(x_k)^2}, \tag{2.3}$$

$$K_n = \sum_{k=1}^{n} \frac{k}{w'(x_k)^2}, \qquad K_n{}^* = \sum_{k=1}^{n} \frac{k(2s_n - x_k)}{w'(x_k)^2}, \tag{2.4}$$

$$F_n = \sum_{k=1}^{n} f(x_k) \frac{w''(x_k)}{w'(x_k)^3}, \qquad F_n{}^* = \sum_{k=1}^{n} f(x_k) \frac{1 + (2s_n - x_k)\, w''(x_k)/w'(x_k)}{w'(x_k)^2}. \tag{2.5}$$

THEOREM 2.1. *Let $h_k(x)$, $h_k{}^*(x)$, $k = 1,\ldots, n$ be as in* (1.2). *Then*

$$\begin{gathered} H_{2n-2}(A_1, f; x) = \sum_{k=1}^{n} f(x_k)\, h_k(x) + c \sum_{k=1}^{n} h_k{}^*(x), \\ c \equiv c_n = F_n/J_n = H'_{2n-2}(A_1, f; x_k), \qquad k = 1,\ldots, n. \end{gathered} \tag{2.6}$$

THEOREM 2.2. *Let $h_k(x)$, $h_k{}^*(x)$, $k = 1,\ldots, n$ be as in* (1.2). *Then*

$$H_{2n-3}(A_2, f; x) = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} (d + ke)\, h_k{}^*(x), \tag{2.7}$$

where

$$d + ke = H'_{2n-3}(A_2, f; x_k), \qquad k = 1,\ldots, n, \tag{2.8}$$

and d, e are given by

$$d \equiv d_n = \frac{-K_n F_n{}^* + K_n{}^* F_n}{J_n K_n{}^* - J_n{}^* K_n}, \qquad e \equiv e_n = \frac{J_n F_n{}^* - J_n{}^* F_n}{J_n K_n{}^* - J_n{}^* K_n}. \tag{2.9}$$

As the proof of Theorem 2.1 is similar to that of Theorem 2.2, we shall prove only the latter.

Proof of Theorem 2.2. By definition, $H_{2n-3}(A_2, f; x)$ satisfies the linear difference equation

$$H'_{2n-3}(A_2, f; x_k) - 2H'_{2n-3}(A_2, f; x_{k+1}) + H'_{2n-3}(A_2, f; x_{k+2}) = 0,$$

whose general solution is given by (2.8), with arbitrary d, e. Since the degree of $H_{2n-3}(A_2, f; x)$ cannot exceed $2n - 3$, we can determine d, e by requiring that the coefficients of x^{2n-1}, x^{2n-2} in the identity

$$H_{2n-3}(A_2, f; x) = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} H'_{2n-3}(A_2, f; x_k)\, h_k{}^*(x) \tag{2.10}$$

should vanish. It is easy to see from (1.2) that

$$h_k(x) = \frac{-w''(x_k)}{w'(x_k)^3} x^{2n-1} + \frac{1 + (2s_n - x_k) w''(x_k)/w'(x_k)}{w'(x_k)^2} x^{2n-2} + \cdots,$$

$$h_k^*(x) = \frac{1}{w'(x_k)^2} x^{2n-1} - \frac{2s_n - x_k}{w'(x_k)^2} x^{2n-2} + \cdots .$$

Therefore, after a brief calculation, we see that the vanishing of the coefficients of x^{2n-1}, x^{2n-2} in (2.10) yields

$$dJ_n + eK_n - F_n = 0, \qquad dJ_n^* + eK_n^* - F_n^* = 0,$$

hence, (2.9). Q.E.D.

Remark 2.1. An interesting property of $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$ is that, unlike the Hermite–Féjer polynomial $H_{2n-1}(f, x)$, they reproduce polynomials up to degree 1, as can be seen directly from the defining conditions (2.1). If the nodes X_n are equidistant, then $H_{2n-3}(A_2, f; x)$ reproduces polynomials up to degree 2.

Remark 2.2. When the nodes X_n are *symmetrical* (i.e., $x_{n+1-k} = -x_k$, $k = 1,..., n$), it is easy to see directly from (2.1) that $H_{2n-1}(f, x)$, $H_{2n-2}(A_1, f; x)$, $H_{2n-3}(A_2, f; x)$ are all even (odd) if $f(x)$ is even (odd). Therefore, it is

$$H_{2n-2}(A_1, f; x) = H_{2n-1}(f, x), \qquad \text{if } f(x) \text{ is even},$$
$$H_{2n-3}(A_2, f; x) = H_{2n-2}(A_1, f; x), \qquad \text{if } f(x) \text{ is odd}.$$

3. Convergence of $H_{2n-1-m}(A_m, f; x)$ Based on The Roots of $P_n^{(\alpha,\beta)}(x)$

Let $\alpha, \beta > -1$, and let $w(x) = P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n defined by the differential equation

$$(1 - x^2) w'' + (\beta - \alpha - (\alpha + \beta + 2) x) w' + n(n + \alpha + \beta + 1) w = 0,$$

with initial condition $w(1) = \binom{n+\alpha}{n}$. From the theory of orthogonal polynomials, we need to recall the relation [9, (8.9.1), p. 236]

$$\arccos x_k = k\pi n^{-1} + O(n^{-1}), \qquad k = 1,..., n, \tag{3.2}$$

and the quadrature formula [9, (15.3.1), p. 349]

$$\int_{-1}^{1} g(x)(1 - x)^\alpha (1 + x)^\beta \, dx = \sum_{k=1}^{n} \mu_k g(x_k) + R_n(g), \tag{3.3}$$

where

$$\mu_k = \mu_{kn}^{(\alpha,\beta)} = t_n(1 - x_k^2)^{-1} (w'(x_k))^{-2}, \tag{3.4}$$

$$t_n = 2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\, \Gamma(n + \beta + 1)/\Gamma(n + 1)\, \Gamma(n + \alpha + \beta + 1), \tag{3.5}$$

and $R_n(g) = o(1)$ whenever $f \in C[-1, 1]$.

THEOREM 3.1. *If* $f \in C[-1, 1]$ *and* $\alpha, \beta > -1$, *then for all* a, b $(1 > a > b > -1)$, *we have*

$$H_{2n-2}(A_1, f; x) \to f(x), \qquad \text{uniformly on } [b, a], \tag{3.6}$$

$$H_{2n-3}(A_2, f; x) \to f(x), \qquad \text{uniformly on } [b, a]. \tag{3.7}$$

Furthermore, if $\alpha < 0$ $(\beta < 0)$, *we can take* $a = 1$ $(b = -1)$.

As the proofs for (3.6) and (3.7) are similar, we give only the latter. The proof depends on the following two lemmas.

LEMMA 3.1. *Let* t_n *be given by* (3.5), *and let* $w(x) = P_n^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$. *If we set* $q(x) = (1 - x)^{\alpha+1}(1 + x)^{\beta+1}$, *then, as n tends to infinity,*

$$J_n \equiv \sum_{k=1}^{n} \frac{1}{w'(x_k)^2} = t_n^{-1} \int_{-1}^{1} q(x)\, dx + o(1), \tag{3.8}$$

$$J_n' \equiv \sum_{k=1}^{n} \frac{x_k}{w'(x_k)^2} = t_n^{-1} \int_{-1}^{1} q(x)\, x\, dx + o(1), \tag{3.9}$$

$$K_n \equiv \sum_{k=1}^{n} \frac{k}{w'(x_k)^2} = n(\pi t_n)^{-1} \int_{-1}^{1} q(x) \arccos x\, dx + o(1), \tag{3.10}$$

$$K_n' \equiv \sum_{k=1}^{n} \frac{k x_k}{w'(x_k)^2} = n(\pi t_n)^{-1} \int_{-1}^{1} q(x)\, x \arccos x\, dx + o(1). \tag{3.11}$$

Proof. It is enough to derive (3.8) and (3.10), since (3.9) and (3.11) are similarly obtained.

(i) To show (3.8), let us take $g(x) = 1 - x^2$ in the quadrature formula (3.3). Since $g(x)$ is continuous, (3.3) yields

$$\int_{-1}^{1} q(x)\, dx = t_n \sum_{k=1}^{n} \frac{1}{w'(x_k)^2} + t_n o(1). \tag{3.12}$$

As by Stirling's asymptotic formula $t_n \to 2^{\alpha+\beta+1}$ (which is not 0), on dividing (3.12) by t_n we get (3.8).

(ii) To prove (3.10), let us observe that, from (3.2), it follows that $k = n\pi^{-1} \arccos x_k + O(1)$. Hence, by (3.8),

$$\sum_{k=1}^{n} \frac{k}{w'(x_k)^2} = n\pi^{-1} \sum_{k=1}^{n} \frac{\arccos x_k}{w'(x_k)^2} + O(1).$$

The rest follows by taking $g(x) = \arccos x$ in (3.3) and repeating the argument of (i). Q.E.D.

LEMMA 3.2. *Let* $f \in C[-1, 1]$ *and let* J_n, J_n^*, K_n, K_n^*, *be given by* (2.3)–(2.5). *If* $\alpha, \beta > -1$, *then*

$$-K_n F_n^* + K_n^* F_n = O(1), \tag{3.13}$$

$$J_n F_n^* - J_n^* F_n = O(1), \tag{3.14}$$

$$(J_n K_n^* - J_n^* K_n)^{-1} = O(n^{-1}). \tag{3.15}$$

Proof. We divide the proof into two parts.

(i) To prove (3.13) and (3.14), let us first observe that (3.2) implies

$$x_k = \cos(k\pi n^{-1} + O(n^{-1})) = \cos(k\pi n^{-1}) + O(n^{-1}). \tag{3.16}$$

Therefore, it follows easily that

$$s_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \cos(k\pi n^{-1}) + O(1) = O(1). \tag{3.17}$$

On using (2.3), (3.8)–(3.11), and (3.17), we obtain easily

$$\begin{aligned} J_n &= O(1), & J_n^* &= 2s_n J_n - J_n' = O(1), \\ K_n &= O(n), & K_n^* &= 2s_n K_n - K_n' = O(n), \end{aligned} \tag{3.18}$$

and since $f(x)$ is continuous and $s_n = O(1)$, $J_n = O(1)$,

$$F_n = O\left(\sum_{k=1}^{n} \left| \frac{w''(x_k)}{w'(x_k)^3} \right|\right), \qquad F_n^* = O(1) + O\left(\sum_{k=1}^{n} \left| \frac{w''(x_k)}{w'(x_k)^3} \right|\right).$$

From (3.1) it follows that $|w''(x_k)/w'(x_k)| \leqslant C/(1 - x_k^2)$ for some C independent of k, n. Therefore, following the argument used to prove Lemma 3.1, we see that

$$\begin{aligned} \sum_{k=1}^{n} \left| \frac{w''(x_k)}{w'(x_k)^3} \right| &\leqslant C \sum_{k=1}^{n} \frac{1}{(1 - x_k^2)(w'(x_k))^2} \\ &= C t_n^{-1} \int_{-1}^{1} (1 - x)^{\alpha} (1 + x)^{\beta} \, dx + o(1) \\ &= O(1). \end{aligned}$$

Therefore, $F_n = O(1)$, $F_n{}^* = O(1)$, which, combined with (3.18), yields (3.13) and (3.14).

(ii) To prove (3.15), let us set

$$I = \int_{-1}^{1} q(x)\, x\, dx \int_{-1}^{1} q(x) \arccos x\, dx - \int_{-1}^{1} q(x)\, dx \int_{-1}^{1} q(x)\, x \arccos x\, dx.$$

We can easily see, on account of (2.3)–(2.5) and Lemma 3.1, that

$$J_n K_n{}^* - J_n{}^* K_n = -J_n K_n{}' + J_n{}' K_n = nI/\pi t_n{}^2 + o(n). \tag{3.19}$$

It is clear from (3.19) that, to prove (3.15), it is enough to show that $I \neq 0$. But, since $q(x) \geqslant 0$ and x, $\arccos x$ are two nonconstant, monotone functions, $I \neq 0$ is a consequence of the so-called Chebyshev inequality in integral form [7, Theorem 4.3, p. 43]. Q.E.D.

Proof of Theorem 3.1. From (2.6), it follows readily that

$$H'_{2n-3}(A_2, f; x_k) = O(d_n) + nO(e_n), \qquad k = 1, \ldots, n,$$

where d_n, e_n are given by (2.7). Therefore, from Lemma 3.2 it now follows that

$$H'_{2n-3}(A_2, f; x_k) = O(1), \qquad k = 1, \ldots, n. \tag{3.20}$$

By [9, Theorem 14.6, p. 338], (3.20) implies $H_{2n-3}(A_2, f; x) \to f(x)$, uniformly on every closed subinterval of $(-1, 1)$ (if $\alpha < 0$, on every subinterval of the form $[-1, a]$; if $\beta < 0$, on every subinterval of the form $[b, 1]$). Q.E.D.

4. The Polynomials $H_{2n-1-m}(A_m, f; x)$ Based on The Roots of $(1 - x_2)T_{n-2}(x)$

It is convenient to consider, as the nodes of interpolation, the zeros of $(1 - x^2)\, T_n(x)$ rather than those of $(1 - x^2)\, T_{n-2}(x)$. That is, we consider the nodes $x_0, \ldots, x_{n+1}$ given by

$$\begin{aligned} x_0 &\equiv x_{0,n+2} = 1, \qquad x_{n+1} \equiv x_{n+1,n+2} = -1, \\ x_k &\equiv x_{k,n+2} = \cos(2k - 1/2n)\, \pi, \qquad k = 1, \ldots, n. \end{aligned} \tag{4.1}$$

As a consequence of this choice, the degree of the three polynomials $H_{2n-1-m}(A_m, f; x)$ $(m = 0, 1, 2)$ previously considered will be increased by 4, and for greater clarity we shall use for them the notations

$$\mathbf{H}_{2n+3}(f, x), \qquad \mathbf{H}_{2n+2}(A_1, f; x), \qquad \mathbf{H}_{2n+1}(A_2, f; x).$$

Since $T_n(x)$ satisfies the equation

$$(1 - x^2)\, T_n'' - xT_n' + n\, T_n = 0,$$

it is easy to see that, if we set $w(x) = (1 - x^2)\, T_n(x)$, then

$$\begin{aligned} w''(x_k)/w'(x_k) &= -3x_k/(1 - x_k^2), \qquad & x_k \neq \pm 1, \\ &= \pm(2n^2 + 1), & x_k = \pm 1, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} (w'(x_k))^2 &= 1/n^2(1 - x_k^2), \qquad & x_k \neq \pm 1, \\ &= 1/4, & x_k = \pm 1. \end{aligned} \tag{4.3}$$

From the known identity $1 = \sum_{k=1}^{n} (1 - xx_k)\, T_n^2(x)/n^2(x - x_k)^2$, we can easily obtain

$$1 = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 - x_k} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 + x_k} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 - x_k^2}. \tag{4.4}$$

Since the points (4.1) are symmetrical, $s_{n+2} = x_0 + \cdots + x_{n+1} = 0$. It is then easy to see, on using (4.2)–(4.4) and taking into account the notational change from n to $n + 2$, that (2.3)–(2.5) become

$$\begin{aligned} \mathbf{J}_{n+2} &= \sum_{k=0}^{n+1} \frac{1}{w'(x_k)^2} = \frac{3}{2}, \\ \mathbf{J}_{n+2}^* &= \sum_{k=0}^{n+1} \frac{-x_k}{w'(x_k)^2} = 0, \end{aligned} \tag{4.5}$$

$$\begin{aligned} \mathbf{K}_{n+2} &= \sum_{k=0}^{n+1} \frac{k + 1}{w'(x_k)^2} = \frac{n + 2}{2}, \\ \mathbf{K}_{n+2}^* &= \sum_{k=0}^{n+1} \frac{-(k + 1)\, x_k}{w'(x_k)^2} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{kx_k}{1 - x_k^2} + \frac{n}{4}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \mathbf{F}_{n+2} = \mathbf{F}_{n+2}(f) &= \sum_{k=0}^{n+1} f(x_k)\, \frac{w''(x_k)}{w'(x_k)^3} \\ &= \frac{-3}{n^2} \sum_{k=1}^{n} \frac{x_k f(x_k)}{(1 - x_k^2)^2} + \frac{n^2}{2} (f(1) - f(-1)), \\ \mathbf{F}_{n+2}^* = \mathbf{F}_{n+2}^*(f) &= \sum_{k=0}^{n+1} \frac{f(x_k)}{w'(x_k)^2} - \mathbf{F}_{n+2}(xf) \\ &= \frac{1}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{1 - x_k^2} + \frac{1}{4} (f(1) - f(-1)) - \mathbf{F}_{n+2}(xf). \end{aligned} \tag{4.7}$$

If we denote by $\mathbf{h}_k(x)$, $\mathbf{h}_k{}^*(x)$, $k = 0,..., n+1$ the polynomials (1.2) based on the points (4.1), then (2.6) is replaced by

$$\mathbf{H}_{2n+2}(A_1, f; x) = \sum_{k=0}^{n+1} f(x_k)\, \mathbf{h}_k(x) + \mathbf{c}_{n+2} \sum_{k=0}^{n+1} \mathbf{h}_k{}^*(x),$$
$$\mathbf{c}_{n+2} = \mathbf{F}_{n+2}/\mathbf{J}_{n+2} = \mathbf{H}'_{2n+2}(A_1, f; x_k), \qquad k = 0,..., n+1. \tag{4.8}$$

Also, (2.7) is replaced by

$$\mathbf{H}_{2n+1}(A_2, f; x) = \sum_{k=0}^{n+1} f(x_k)\, \mathbf{h}_k(x) + \sum_{k=0}^{n+1} (\mathbf{d}_{n+2} + (k+1)\, \mathbf{e}_{n+2})\, \mathbf{h}_k{}^*(x), \tag{4.9}$$

where

$$\mathbf{d}_{n+2} + (k+1)\, \mathbf{e}_{n+2} = \mathbf{H}'_{2n+1}(A_2, f; x_k), \qquad k = 0,..., n+1, \tag{4.10}$$

and $\mathbf{d}_{n+2}$, $\mathbf{e}_{n+2}$ are given by

$$\mathbf{d}_{n+2} = \frac{-\mathbf{K}_{n+2}\mathbf{F}^*_{n+2} + \mathbf{K}^*_{n+2}\mathbf{F}_{n+2}}{\mathbf{J}_{n+2}\mathbf{K}^*_{n+2} - \mathbf{J}^*_{n+2}\mathbf{K}_{n+2}} = -\frac{n+2}{2}\, \mathbf{e}_{n+2} + \mathbf{c}_{n+2},$$
$$\mathbf{e}_{n+2} = \frac{\mathbf{J}_{n+2}\mathbf{F}^*_{n+2} - \mathbf{J}^*_{n+2}\mathbf{F}_{n+2}}{\mathbf{J}_{n+2}\mathbf{K}^*_{n+2} - \mathbf{J}^*_{n+2}\mathbf{K}_{n+2}} = \frac{\mathbf{F}^*_{n+2}}{\mathbf{K}^*_{n+2}}. \tag{4.11}$$

5. Convergence of $\mathbf{H}_{2n+3}(f, x)$

In this section, we consider the Hermite–Féjer polynomials $\mathbf{H}_{2n+3}(f, x)$ based on the points (4.1).

Our first theorem completely characterizes the uniform convergence class of $\mathbf{H}_{2n+3}(f, x)$, thus adding to the results found earlier by Berman [1, 2]. We recall that $\mathbf{H}_{2n+3}(f, x)$ satisfies

$$\mathbf{H}_{2n+3}(f, x_k) = f(x_k), \qquad \mathbf{H}'_{2n+3}(f, x_k) = 0, \qquad k = 0,..., n+1. \tag{5.1}$$

THEOREM 5.1. *If $f \in C[-1, 1]$, the following three conditions are equivalent*:

$$\mathbf{H}_{2n+3}(f, x) \to f(x), \qquad \textit{uniformly on } [-1, 1], \tag{5.2}$$

$$2n^2(H_{2n-1}(f, \pm 1) - f(\pm 1)) - H'_{2n-1}(f, \pm 1) = o(1), \tag{5.3}$$

$$\frac{1}{n^2} \sum_{k=1}^{n} \frac{f(\pm 1) - f(x_k)}{(1 \mp x_k)^2} = o(1). \tag{5.4}$$

Here, (5.3) and (5.4) consist each of two separate conditions while

$H_{2n-1}(f, x)$ denotes, as before, the Hermite–Féjer polynomial of degree $2n - 1$ defined by

$$H_{2n-1}(f, x_k) = f(x_k), \qquad H'_{2n-1}(f, x_k) = 0, \qquad k = 1,..., n. \tag{5.5}$$

Proof. (i) (5.2) $\Leftrightarrow$ (5.3). Set

$$f_1(x) = \tfrac{1}{2}(f(x) + f(-x)), \qquad f_2(x) = \tfrac{1}{2}(f(x) - f(-x)), \tag{5.6}$$

so that

$$\mathbf{H}_{2n+3}(f, x) = \mathbf{H}_{2n+3}(f_1, x) + \mathbf{H}_{2n+3}(f_2, x).$$

Since $\mathbf{H}_{2n+3}(f, x) \equiv \sum_{k=0}^{n+1} f(x_k)\, \mathbf{h}_k(x)$ and $\mathbf{h}_{n+1-k}(x) = \mathbf{h}_k(-x)$ by symmetry of (4.1), it is easy to see that

$$\begin{aligned} \mathbf{H}_{2n+3}(f_1, x) &= \tfrac{1}{2}(\mathbf{H}_{2n+3}(f, x) + \mathbf{H}_{2n+3}(f, -x)), \\ \mathbf{H}_{2n+3}(f_2, x) &= \tfrac{1}{2}(\mathbf{H}_{2n+3}(f, x) - \mathbf{H}_{2n+3}(f, -x)). \end{aligned} \tag{5.7}$$

Therefore, it is enough to prove the equivalence of (5.2) and (5.3) when $f(x) = f_1(x)$ and $f(x) = f_2(x)$. We limit ourselves to the latter case, the proof of the former being similar.

From (5.5) and (5.7) it is clear that $\mathbf{H}_{2n+3}(f_2, x)$ and $H_{2n-1}(f_2, x)$ are odd. Therefore, we can write, on account of (5.1) and (5.5),

$$\mathbf{H}_{2n+3}(f_2, x) - H_{2n-1}(f_2, x) = (p_n x + q_n x^3)\, T_n^2(x), \tag{5.8}$$

where p_n, q_n depend on $f(x)$ and on n. Since $H_{2n-1}(f, x) \to f(x)$ uniformly on $[-1, 1]$ by Féjer's result [3], we immediately see from (5.8) that (5.2) is equivalent to the simultaneous occurrence of $p_n \to 0$ and $q_n \to 0$. Evaluating (5.8) and its derivative at $x = 1$ we obtain, on simplifying by means of (5.1),

$$\begin{aligned} p_n + q_n &= f_2(1) - H_{2n-1}(f_2, 1), \\ p_n + 3q_n &= -2n^2(f_2(1) - H_{2n-1}(f_2, 1)) - H'_{2n-1}(f_2, 1). \end{aligned}$$

Since $p_n + q_n \to 0$ by Féjer's result, the condition $p_n \to 0$, $q_n \to 0$ is then equivalent to $p_n + 3q_n \to 0$, hence, as $f_2(1) = -f_2(-1)$, it is equivalent to (5.2).

(ii) (5.3) $\Leftrightarrow$ (5.4). Here we assume that $f(x)$ is completely arbitrary. It is sufficient to prove that

$$2n^2(H_{2n-1}(f, 1) - f(1)) - H'_{2n-1}(f, 1) = o(1)$$

is equivalent to

$$\sum_{k=1}^{n} \frac{f(1) - f(x_k)}{(1 - x_k)^2} = o(n^2),$$

as the proof for the point $x = -1$ is quite similar. Differentiating the known formula $H_{2n-1}(f, x) \equiv \sum_{k=1}^{n} f(x_k)(1 - xx_k) T_n^2(x)/n^2(x - x_k)^2$ and simplifying by means of $T_n(1) = 1$, $T_n'(1) = 0$ we get

$$H'_{2n-1}(f, 1) = \frac{-3}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1 - x_k)^2},$$

and hence,

$$2n^2(H_{2n-1}(f, 1) - f(1)) - H'_{2n-1}(f, 1)$$

$$= 2 \sum_{k=1}^{n} \frac{f(1)}{1 - x_k} - \frac{3}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1 - x_k)^2}. \tag{5.9}$$

If $f(x) \equiv 1$, then $H_{2n-1}(f, x) \equiv 1$ and the left-hand side of (5.9) vanishes. This yields

$$\sum_{k=1}^{n} \frac{2}{1 - x_k} = \frac{3}{n^2} \sum_{k=1}^{n} \frac{1}{(1 - x_k)^2}. \tag{5.10}$$

Thus (5.9) can be further simplified to

$$2n^2(H_{2n-1}(f, 1) - f(1)) - H'_{2n-1}(f, 1) = \frac{-3}{n^2} \sum_{k=1}^{n} \frac{f(1) - f(x_k)}{(1 - x_k)^2}.$$

This completes the proof of Theorem 5.1. Q.E.D.

A simpler sufficient condition for $\mathbf{H}_{2n+3}(f, x)$ to converge uniformly to $f(x)$ is given by:

THEOREM 5.2. *If $f \in C[-1, 1]$ is differentiable at $x = \pm 1$ and*

$$f'(1) = f'(-1) = 0, \tag{5.11}$$

then $\mathbf{H}_{2n+3}(f, x) \to f(x)$ uniformly on $[-1, 1]$.

Proof. After Theorem 5.1, it is enough to show that (5.11) implies (5.4). We only consider the case $x = 1$, the proof for $x = -1$ being similar. For arbitrary δ ($0 < \delta < 1$), we have

$$\left| \sum_{k=1}^{n} (f(1) - f(x_k))/(1 - x_k)^2 \right| \leqslant | \Sigma_{|1-x_k|>\delta} | + | \Sigma_{|1-x_k|\leqslant\delta} | = I_1 + I_2. \tag{5.12}$$

Since $f'(1)$ exits, we immediately obtain

$$I_1 \leqslant (1/\delta) \sum_{k=1}^{n} |(f(1) - f(x_k))/(1 - x_k)| \leqslant C(n/\delta),$$

for some C independent of n. Also, as $1 - x_k > 0$, we see that

$$I_2 \leqslant \max_{|1-x_k| \leqslant \delta} |(f(1) - f(x_k))/(1 - x_k)| \sum_{k=1}^{n} (1/(1 - x_k)).$$

Using (4.4) and (5.11) we thus obtain

$$I_1 + I_2 \leqslant Cn\delta^{-1} + n^2\epsilon(\delta),$$

where $\epsilon(\delta) \to 0$ if $\delta \to 0$. Taking $\delta = 1/\log n$ and using (5.12) we get (5.4).

Q.E.D.

Remark 5.1. Since

$$C_1k^2/n^2 \leqslant 1 - x_k = 2\sin^2(2k - 1/4n)\pi \leqslant C_2k^2/n^2, \qquad k = 1,..., n, \tag{5.13}$$

where C_1, C_2 are independent of k, n, it is easy to see that, if $f \in C[-1, 1]$ has nonzero derivative (finite or not) at $x = 1$, then

$$\left| \sum_{k=1}^{n} (f(1) - f(x_k))/(1 - x_k)^2 \right| \geqslant C_3n^2,$$

with C_3 independent of n. Hence, by Theorem 5.1, $\mathbf{H}_{2n+3}(f, x)$ does not converge uniformly to $f(x)$. In particular, this happens when $f(x) = x$, x, or x^2, in agreement with Berman's results [1, 2].

6. Convergence of $\mathbf{H}_{2n+2}(A_1, f; x)$ and $\mathbf{H}_{2n+1}(A_2, f: x)$

Here, we give a partial characterization of the uniform convergence classes of the averaging Hermite interpolators (4.8) and (4.9).

Theorem 6.1. *If $f \in C[-1, 1]$ and its even and odd parts $f_1(x)$, $f_2(x)$ defined in* (5.6) *satisfy*

$$(1/n^2) \sum_{k=1}^{n} (f_1(1) - f_1(x_k))/(1 - x_k)^2 = o(1), \tag{6.1}$$

$$(1/n^2) \sum_{k=1}^{n} (f_2(1) - f_2(x_k))/(1 - x_k)^2 = o(n), \tag{6.2}$$

then $\mathbf{H}_{2n+2}(A_1, f; x) \to f(x)$ uniformly on $[-1, 1]$. Furthermore, the condition $o(1)$ cannot be replaced by $O(1)$, nor $o(n)$ by $O(n)$.

A simpler characterization is provided by

THEOREM 6.2. *If* $f \in C[-1, 1]$ *and its even and odd parts* $f_1(x)$, $f_2(x)$ *defined in* (5.6) *satisfy*

$$f_1(x) \text{ is differentiable at } x = \pm 1 \quad \text{and} \quad f_1'(1) = f_1'(-1) = 0, \quad (6.3)$$

$$f_2(1) - f_2(x) = o((1 - x)^{1/2}), \qquad x \to 1-, \quad (6.4)$$

then $\mathbf{H}_{2n+2}(A_1, f; x) \to f(x)$ *uniformly on* $[-1, 1]$. *Furthermore, the condition* $o((1 - x)^{1/2})$ *cannot be replaced by* $O((1 - x)^{1/2})$.

After Remark 2.2, the two conditions (6.1) and (6.3) follow immediately from conditions (5.4) of Theorem 5.1 and condition (5.11) of Theorem 5.2. Similarly, (6.2) follows from condition (6.5) of Theorem-6.3, while (6.4) follows from condition (6.6) of Theorem 6.4.

THEOREM 6.3. *If* $f \in C[-1, 1]$ *satisfies*

$$(1/n^2) \sum_{k=1}^{n} (f(\pm 1) - f(x_k))/(1 \mp x_k)^2 = o(n), \quad (6.5)$$

then $\mathbf{H}_{2n+1}(A_2, f; x) \to f(x)$ *uniformly on* $[-1, 1]$. *Furthermore, the condition* $o(n)$ *cannot be replaced by* $O(n)$.

THEOREM 6.4. *If* $f \in C[-1, 1]$ *satisfies*

$$f(\pm 1) - f(x) = o((1 \mp x)^{1/2}), \qquad x \to \pm 1 \mp, \quad (6.6)$$

then $\mathbf{H}_{2n+1}(A_2, f; x) \to f(x)$ *uniformly on* $[-1, 1]$. *Furthermore, the condition* $o((1 \mp x)^{1/2})$ *cannot be replaced by* $O((1 \mp x)^{1/2})$.

Note that both (6.5) and (6.6) consists of two separate conditions to hold simultaneously. The proof depends on the three lemmas below.

LEMMA 6.1. *Let* $h_k^*(x)$, $k = 1, \ldots, n$ *be the polynomials* (1.2) *based on the zeros* $x_1, \ldots, x_n$ *of* $T_n(x)$. *Then*

$$\sum_{k=1}^{n} h_k^*(x) = T_n(x)\, T_{n-1}(x)/n, \quad (6.7)$$

$$\sum_{k=1}^{n} k h_k^*(x) = O(1), \qquad x \in [-1, 1]. \quad (6.8)$$

Proof. (i) Formula (6.7) is due to Féjer [4, (59), p. 300].

(ii) From the explicit expression (4.1) of $x_1, \ldots, x_n$, we obtain

$$k = ((n+1)/2) - (n/\pi) \arcsin x_k, \qquad k = 1, \ldots, n.$$

Therefore, it follows from (6.7) that (6.8) is equivalent to

$$S_n(x) \equiv \sum_{k=1}^{n} (\arcsin x - \arcsin x_k)\, h_k^*(x) = O(n^{-1}), \qquad x \in [-1, 1]. \quad (6.9)$$

To prove (6.9), let first observe that, on account of the explicit expression $h_k^*(x) = (1 - x_k^2)\, T_n^2(x)/(x - x_k)\, n^2$, $S_n(x)$ becomes

$$S_n(x) = \left(\frac{T_n(x)}{n}\right)^2 \sum_{k=1}^{n} \frac{\arcsin x - \arcsin x_k}{x - x_k} (1 - x_k^2). \quad (6.10)$$

The function $\arcsin x$ is increasing, odd, convex on [0, 1] and concave on [−1, 0]. It is then easy to see geometrically that, for all $x \in [0, 1]$,

$$0 \leqslant \frac{\arcsin x - \arcsin x_k}{x - x_k} \leqslant \frac{\pi/2 - \arcsin x_k}{1 - x_k}, \qquad k = 1, \ldots, n.$$

Using this in (6.10) we obtain, after simplification, the inequalities

$$0 \leqslant S_n(x) \leqslant T_n^2(x)\, n^{-2} \sum_{k=1}^{n} (\pi/2 - \arcsin x_k)(1 + x_k) \leqslant 2\pi/n, \quad (6.11)$$

valid on [0, 1]. Similarly, we can see that on [−1, 0], the inequalities

$$0 \leqslant S_n(x) \leqslant T_n^2(x)\, n^{-2} \sum_{k=1}^{n} (\pi/2 + \arcsin x_k)(1 - x_k) \leqslant 2\pi/n \quad (6.12)$$

are valid. Combining (6.11) and (6.12), we obtain (6.9) and thus (6.8). Q.E.D.

LEMMA 6.2. *Let* $\mathbf{c}_{n+2}$ *be as in* (4.8) *and let* $h_k(x)$, $h_k^*(x)$, $k = 1, \ldots, n$ *be the polynomials* (1.2) *based on the zeros of* $T_n(x)$. *If* $f \in C[-1, 1]$ *is odd and satisfies* (6.5), *then*

$$\mathbf{c}_{n+2} \sum_{k=1}^{n} h_k^*(x) = o(1), \qquad x \in [-1, 1]. \quad (6.13)$$

Proof. On account of (6.7), it is enough to show that, if $f(x)$ is odd, then (6.5) implies $\mathbf{c}_{n+2} = o(n)$. From (4.5) and (4.7), it follows that $\mathbf{c}_{n+2} = 2\mathbf{F}_{n+2}(f)/3$. Since $f(x)$ is odd and the points $x_1, \ldots, x_n$ of (4.1) are symmetrical, we have

$$\frac{1}{n^2} \sum_{k=1}^{n} \frac{2x_k f(x_k)}{(1 - x_k^2)^2} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{(1 + x_k)^2 f(x_k)}{(1 - x_k)^2 (1 + x_k)^2} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1 - x_k)^2}. \quad (6.14)$$

From (4.4) and (5.10), it follows that

$$\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{(1-x_k)^2}=\frac{2n^2}{3}. \tag{6.15}$$

Now, multiply (6.15) by $3(f(1)-f(-1))/4=3f(1)/2$, (6.14) by -3, and add together. Recalling that $\mathbf{c}_{n+2}=2\mathbf{F}_{n+2}(f)/3$, we then have

$$\mathbf{c}_{n+2}=\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2}. \tag{6.16}$$

It is now clear that (6.5) implies $\mathbf{c}_{n+2}=o(n)$, and hence, (6.13). Q.E.D.

LEMMA 6.3. *Let* $\mathbf{d}_{n+2}$, $\mathbf{e}_{n+2}$ *be as in* (4.11) *and let* $h_k(x)$, $h_k^*(x)$, $k=1,\ldots,n$ *be the polynomials* (1.2) *based on the zeros of* $T_n(x)$. *If* $f\in C[-1,1]$ *is even and satisfies* (6.5), *then*

$$\sum_{k=1}^{n}(\mathbf{d}_{n+2}+(k+1)\,\mathbf{e}_{n+2})\,h_k^*(x)=o(1),\qquad x\in[-1,1]. \tag{6.17}$$

Proof. On account of (6.7) and (6.8), it is enough to show that, if $f(x)$ is even, then (6.5) implies $\mathbf{d}_{n+2}=o(n)$ and $\mathbf{e}_{n+2}=o(1)$. As the points $x_1,\ldots,x_n$ are symmetrical and $f(x)$ is even, $\mathbf{c}_{n+2}$ vanishes, hence, the two conditions we have to prove reduce to $\mathbf{e}_{n+2}=o(1)$.

Since $f(x)$ is even and continuous, it follows from (4.4) and (4.7) that

$$\mathbf{F}_{n+2}^*(f)=\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k^2}-\mathbf{F}_{n+2}(xf)=O(1)-\mathbf{F}_{n+2}(xf).$$

Replacing in (6.16) the function $f(x)$ by $xf(x)$ and using (4.4), we get

$$\begin{aligned}\mathbf{F}_{n+2}(xf)&=\frac{3}{2n^2}\sum_{k=1}^{n}\frac{f(1)-x_kf(x_k)}{(1-x_k)^2}\\&=\frac{3}{2n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2}-\frac{3}{2n^2}\sum_{k=1}^{n}\frac{(1-x_k)f(x_k)}{(1-x_k)^2}\\&=\frac{3}{2n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2}+O(1).\end{aligned}$$

It is clear, therefore, that (6.5) implies $\mathbf{F}_{n+2}(xf)=o(n)$. Now, from (5.13), we can see that

$$\frac{1}{n^2}\sum_{k=1}^{n}\frac{kx_k}{1-x_k^2}=\frac{1}{2n^2}\sum_{k=1}^{n}\left(\frac{k}{1-x_k}+\frac{k}{1+x_k}\right)=O(\log n).$$

Thus, on account of (4.6) and (4.11), (6.5) implies $\mathbf{e}_{n+2}=o(1)$. Q.E.D.

Proof of Theorem 6.3. We prove the two parts of the theorem separately.

(i) Let $h_k(x)$, $h_k{}^*(x)$, $k = 1,..., n$ be the polynomials (1.2) based on the zeros $x_1, ..., x_n$ of $T_n(x)$, and set

$$H_{2n-1}^*(f, x) = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} \mathbf{H}'_{2n+1}(A_2, f, x_k)\, h_k{}^*(x). \quad (6.18)$$

From Féjer's result [3], it follows that since $f \in C[-1, 1]$,

$$H_{2n-1}^*(f, x) = f(x) + \sum_{k=1}^{n} \mathbf{H}'_{2n+1}(A_2, f; x_k)\, h_k{}^*(x) + o(1), \qquad x \in [-1, 1]. \quad (6.19)$$

Since $H_{2n-1}^*(f, x)$ and $\mathbf{H}_{2n+1}(A_2, f; x)$, as well as their derivatives, coincide at $x_1, ..., x_n$, we can write for some p_n, q_n,

$$\mathbf{H}_{2n+1}(A_2, f; x) - H_{2n-1}^*(f, x) = (p_n + q_n x)\, T_n{}^2(x). \quad (6.20)$$

As $\mathbf{H}_{2n+1}(A_2, f; \pm 1) = f(\pm 1)$, on setting $x = \pm 1$, we get

$$\begin{aligned} p_n &= \tfrac{1}{2}(f(1) + f(-1)) - \tfrac{1}{2}(H_{2n-1}^*(f, 1) + H_{2n-1}^*(f, -1)), \\ q_n &= \tfrac{1}{2}(f(1) - f(-1)) - \tfrac{1}{2}(H_{2n-1}^*(f, 1) - H_{2n-1}^*(f, -1)). \end{aligned} \quad (6.21)$$

It is clear, therefore, from (6.19)–(6.21), that $\mathbf{H}_{2n+1}(A_2, f; x) \to f(x)$ uniformly on $[-1, 1]$ if

$$\sum_{k=1}^{n} \mathbf{H}'_{2n+1}(A_2, f; x_k)\, h_k{}^*(x) = o(1), \qquad x \in [-1, 1]. \quad (6.22)$$

If we now separate $f(x)$ into its even and odd parts and take (4.8) and (4.11) into account, we find that (6.22) follows from Lemmas 6.2 and 6.3.

(ii) To prove the second half of Theorem 6.3, we show that $f(x) = x(1 - x^2)^{1/2}$ satisfies

$$\frac{1}{n^2} \sum_{k=1}^{n} \frac{f(1) - f(x_k)}{(1 - x_k)^2} = O(n), \qquad \frac{1}{n^2} \sum_{k=1}^{n} \frac{f(-1) - f(x_k)}{(1 + x_k)^2} = O(n), \quad (6.23)$$

while $\mathbf{H}_{2n+1}(A_2, f; x)$ fails to converge to $f(x)$ uniformly on $[-1, 1]$.

To see (6.23), let us observe first that, since $f(x)$ is odd, the two sums in (6.23) are equal and, on account of (6.16) and Remark 2.2, their common value is $\mathbf{c}_{n+2}$, given by (4.8). Using the symmetry of $x_1, \ldots, x_n$ and the inequalities (5.13), we see that

$$\begin{aligned}\mathbf{c}_{n+2} &= \frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2} = \frac{1}{n^2}\sum_{k=1}^{[n/2]}\frac{x_k(1-x_k^2)^{1/2}}{(1-x_k)^2} \\ &= \frac{1}{n^2}\sum_{k=1}^{[n/2]}\frac{x_k(1+x_k)^{1/2}}{(1-x_k)^{3/2}} \leqslant \frac{2}{n^2}\sum_{k=1}^{[n/2]}\frac{1}{(1-x_k)^{3/2}} \leqslant Cn,\end{aligned} \tag{6.24}$$

for some constant $C > 0$. On the other hand, it is clear that, for some other constant $c > 0$,

$$\mathbf{c}_{n+2} \geqslant x_1(1+x_1)^{1/2}/n^2(1-x_1)^{3/2} \geqslant cn. \tag{6.25}$$

To see that $\mathbf{H}_{2n+1}(A_2, f; x)$ does not converge to $f(x)$ uniformly on $[-1, 1]$, let us first note that, on account of (6.7), (6.19) may be written

$$H_{2n-1}^*(f, x) = f(x) + n^{-1}\mathbf{c}_{n+2}T_n(x)\,T_{n-1}(x) + o(1), \qquad x \in [-1, 1]. \tag{6.26}$$

Replacing (6.26) into (6.21), we get $p_n = o(1)$, $q_n = n^{-1}\mathbf{c}_{n+2} + o(1)$. Using this and (6.26) to simplify (6.20), we obtain

$$\mathbf{H}_{2n+1}(A_2, f; x) - f(x) = n^{-1}\mathbf{c}_{n+2}(xT_n^2(x) - T_n(x)\,T_{n-1}(x)) + o(1), \qquad x \in [-1, 1].$$

Setting $x = \cos t$, it is easy to see that

$$U_n(x) = xT_n^2(x) - T_n(x)\,T_{n-1}(x) = -\tfrac{1}{2}\sin t \sin 2nt. \tag{6.27}$$

It is clear that, on any subinterval I of $[-1, 1]$, we have, at least for all suitably large n, $\sup_I |U_n(x)| \geqslant \frac{1}{4}\sup_I |\sin t|$. Thus, it follows from (6.25) that $n^{-1}\mathbf{c}_{n+2}|U_n(x)| \geqslant C > 0$, for some constant C. Therefore, $\mathbf{H}_{2n+1}(A_2, f; x)$ fails to converge to $f(x)$ uniformly on any subinterval I of $[-1, 1]$. Q.E.D.

Proof of Theorem 6.4. The second part has just been established in part (ii) above.

To prove the first part, we show that (6.6) implies (6.5). To see this, let observe that, much as in the proof of Theorem 5.2, we have, for arbitrary δ $(0 < \delta < 1)$,

$$\left|\sum_{k=1}^{n}(f(1) - f(x_k))/(1-x_k)^2\right| \leqslant C(n/\delta^2) + n^2 \max_{|1-x_k|\leqslant\delta} |(f(1) - f(x_k))/(1-x_k)|.$$

On account of (6.6), it is easy to see that

$$\begin{aligned}\left|(1/n^2)\sum_{k=1}^{n}(f(1)-f(x_k))/(1-x_k)^2\right| \\ \leqslant (C/n\delta^2) + \epsilon(\delta)\max_{|1-x_k|\leqslant\delta}(1-x_k)^{-1/2} \\ = (C/n\delta^2) + (\epsilon(\delta)/(1-x_1)^{1/2}) \leqslant (C/n\delta^2) + C_1 n\epsilon(\delta),\end{aligned}$$

where $\epsilon(\delta)\to 0$ if $\delta\to 0$. Taking $\delta = (\log n/n)^{1/2}$, we obtain one half of (6.5). The other half can be similarly obtained. Q.E.D.

7. Uniform Convergence Classes of $\mathbf{H}_{2n+3-m}(A_m, f; x)$

Let $\mathbf{H}_{2n+3}^c$ denote the uniform convergence class of $\mathbf{H}_{2n+3}(f, x)$, i.e., let

$$\mathbf{H}_{2n+3}^c = \{f\in C[-1,1]: \mathbf{H}_{2n+3}(f,x)\to f(x) \text{ uniformly on } [-1,1]\},$$

and let $\mathbf{H}_{2n+2}^c(A_1)$ and $\mathbf{H}_{2n+1}^c(A_2)$ be analogously defined.

Theorem 7.1. *We have, with strict inclusion,*

$$\mathbf{H}_{2n+3}^c \subset \mathbf{H}_{2n+2}^c(A_1) \subset \mathbf{H}_{2n+1}^c(A_2). \tag{7.1}$$

Proof. It is easy to see, on account of Remark 2.2, that if $f(x)$ is even (odd), that $f\in \mathbf{H}_{2n+3}^c$ implies $f\in\mathbf{H}_{2n+2}^c(A_1)$. If $f(x)$ is arbitrary, we arrive at the same conclusion by separating $f(x)$ into its even and odd parts and using linearity. This establishes one half of (7.1); the other half is similarly obtained.

To see that the inclusions in (7.1) are strict, it is enough to exhibit functions that are in one class but not in the other.

(i) $\mathbf{H}_{2n+3}^c \neq \mathbf{H}_{2n+2}^c(A_1)$. In fact, $f(x) = x$ is in $\mathbf{H}_{2n+2}^c(A_1)$ on account of Theorem 6.2, but not in $\mathbf{H}_{2n+3}^c$ by Remark 5.1 (see also [1, 2]).

(ii) $\mathbf{H}_{2n+2}^c(A_1) \neq \mathbf{H}_{2n+1}^c(A_2)$. Consider $f(x) = x^2$. Since $f(x)$ is even, we have $\mathbf{H}_{2n+2}(A_1, f; x) = \mathbf{H}_{2n+3}(f, x)$ on account of Remark 2.2. Since $f'(1) = 1$, we see from Remark 5.1 (or from [1, 2]) that $\mathbf{H}_{2n+3}(f, x)$ fails to converge uniformly to $f(x)$ on $[-1, 1]$. Therefore, $f(x)$ does not belong to $\mathbf{H}_{2n+2}^c(A_1)$. However, on account of Theorem 6.4, $f(x)$ is obviously in $\mathbf{H}_{2n+1}^c(A_2)$. Q.E.D.

8. Conclusions

It is apparent, from Theorem 3.1 and Theorem 7.1, that the Hermite–Féjer operator $H_{2n-1}(f, x)$ and the averaging Hermite interpolators $H_{2n-2}(A_1, f; x)$ and $H_{2n-3}(A_2, f; x)$ yield three sequences of polynomial operators of increasing power, at least in the cases when $w(x) = P_n^{(\alpha,\beta)}(x)$ and $w(x) = (1 - x^2)\, T_{n-2}(x)$. It would be interesting to see whether or not the averaging Hermite interpolators $H_{2n-1-m}(A_m, f; x)$ with $m \geqslant 3$ give yet more powerful operators. For instance, one may consider the operator $H_{2n-5}(A_4, f; x)$, where $A_4(z) = (1 - z)^4$, and ask whether the uniform convergence classes in the cases $w(x) = P_n^{(\alpha,\beta)}(x)$ and $w(x) = (1 - x^2)T_{n-2}(x)$ are the full $C[-1, 1]$.

Acknowledgments

The author wishes to thank Professor A. Sharma and Dr. J. Tzimbalario for their useful suggestions.

References

1. D. L. Berman, On the theory of interpolation, *Soviet Math. Dokl.* **6** (1965), 945–948; transl. from *Dokl. Akad. Nauk SSSR* **163** (1965), 551–554.
2. D. L. Berman, A study of the Hermite–Féjer interpolation process, *Soviet Math. Dokl.* **10** (1969), 813–816; transl. from *Dokl. Akad. Nauk SSSR* **187** (1969), 241–244).
3. L. Féjer, Die Abschätzung eines Polynoms in einem Intervalle... *Math. Z.* **32** (1930), 426–457; also [4, pp. 285–317].
4. L. Féjer, "Gesammelte Arbeiten," II. Budapest, 1970.
5. M. A. Botto and A. Sharma, Averaging interpolation on sets with multiplicities, *Aequationes Math.* to appear.
6. G. Grünwald, On the theory of interpolation, *Acta Math. Acad. Sci. Hungar.* **75** (1942), 219–245.
7. G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities," Cambridge, Univ. Press, London/New York, 1964.
8. G. Szegö, Über gewisse Interpolationspolynome die zu den Jacobischen und Laguerreschen Abzissen gehören, *Math. Z.* **35** (1932), 579–602; also [9, Theorem 14.6, p. 338].
9. G. Szegö, Orthogonal Polynomials," Vol. 23, Amer. Math. Soc. Colloquium Publications, Providence, R.I., 1959; revised ed. 1967.